

Transport Coefficients of Bulk Viscous Pressure in the 14-moment approximation

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We compute the transport coefficients that appear in the fluid-dynamical equations for the bulk viscous pressure and shear-stress tensor using the 14-moment approximation in the limit of small, but finite, masses. In this limit, we are able to express all these coefficients in terms of known thermodynamic quantities, such as the thermodynamic pressure, energy density, and the velocity of sound. We explicitly demonstrate that the ratio of bulk viscosity to bulk relaxation time behaves very differently, as a function of temperature, than the ratio of shear viscosity to shear relaxation time. We further explicitly compute, for the first time, the transport coefficients that couple the bulk viscous pressure to the shear-stress tensor and vice versa. The coefficient that couples bulk viscous pressure to shear-stress tensor is found to be orders of magnitude larger than the bulk viscosity itself, suggesting that bulk viscous pressure production owes more to this coupling than to the expansion rate of the system.

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I. INTRODUCTION

The effect of dissipation on the dynamics of the quark-gluon plasma (QGP) created in ultrarelativistic heavy ion collisions has been widely investigated by several authors [1]. In the last 5 years, most of the effort on this topic was concentrated in extracting the magnitude of the shear viscosity of the QGP from experiment, i.e., in investigating how small the shear viscosity to entropy density ratio actually is [2–14]. In parallel, while attempts were made to understand the behavior of the bulk viscosity of hot and dense nuclear matter in general [15–26], and while some simulations of relativistic heavy ion collisions did consider the effects of bulk viscous pressure on the dynamics of the QGP [27–36], it is fair to write that the effects of bulk viscosity still remain to be studied systematically in numerical simulations of heavy-ion collisions.

At very high temperatures, it is established that bulk viscosity of QCD matter is much smaller than the shear viscosity, which led to the belief that bulk viscosity would play a much smaller role when compared to shear viscosity in the description of the bulk nuclear matter created in heavy ion collisions. However, one should still note that, in the temperature region produced experimentally in heavy ion collisions, the actual order of magnitude and temperature dependence of bulk viscosity is unknown [37] and, in principle, it can be large enough to affect the time evolution of bulk QCD matter.

When it comes to the behavior of the remaining transport coefficients that appear in the equation of motion for the bulk viscous pressure, the uncertainties are even larger. Basic features such as the behavior of the bulk relaxation time, how the bulk viscous pressure couples to the shear-stress tensor, among others, are not well understood even on a qualitative level. Together with uncertainties on how to implement the freezeout procedure when bulk viscous pressure is present [36, 38, 39], this makes it more complicated to establish concrete results about the influences of bulk viscosity on heavy ion collision observables on a phenomenological level.

Therefore, it is relevant to study the aforementioned uncertainties first in simpler systems, where they can be analyzed in detail. For instance, these questions can be addressed in the framework of the relativistic Boltzmann equation, where all transport coefficients that appear in the equations of relativistic fluid dynamics are in principle known and can be explicitly computed. Similar investigations were already done for shear-stress tensor and heat flow [40].

The equations of motion and transport coefficients of a relativistic fluid can be derived from the relativistic Boltzmann equation using the 14-moment approximation [41–43]. The main equations of motion are the conservation laws of energy, momentum, and charge, which follow directly from the conservation of these quantities on a microscopic level. For the sake of simplicity, in this paper we will always assume that the net charge is always zero and that no charge diffusion takes place. Then, we only have to solve the continuity equations for the energy-momentum tensor, $T^{\mu\nu}$,

$$\partial_\mu T^{\mu\nu} = 0 .$$

The above equation is general and independent of the formalism employed to derive fluid dynamics. The remaining 6 equations of motion that close this system (9 equations, if one takes into account charge diffusion), i.e., the time

evolution equations for the shear-stress tensor and bulk viscous pressure, are less general and must be derived within a certain framework. In the case of the Boltzmann equation, a possible framework is the already mentioned 14-moment approximation developed by Israel and Stewart (in principle, it should be called 10-moment approximation, since we don't include net charge density and diffusion 4-current, but we shall keep naming it 14-moment nevertheless). In the case where only bulk viscous pressure and shear-stress tensor are present, the 14-moment approximation leads to the following relaxation-type equations [42, 43]

$$\tau_{\Pi} \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \varphi_1\Pi^2 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} + \varphi_3\pi^{\mu\nu}\pi_{\mu\nu}, \quad (1)$$

$$\begin{aligned} \tau_{\pi} \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = & 2\eta\sigma^{\mu\nu} + 2\pi_{\alpha}^{\langle\mu}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \varphi_7\pi_{\alpha}^{\langle\mu}\pi^{\nu\rangle\alpha} - \tau_{\pi\pi}\pi_{\alpha}^{\langle\mu}\sigma^{\nu\rangle\alpha} \\ & + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + \varphi_6\Pi\pi^{\mu\nu}. \end{aligned} \quad (2)$$

Here, Π is the bulk viscous pressure and $\pi^{\mu\nu}$ is the shear-stress tensor. We further introduced the vorticity tensor $\omega_{\lambda\rho} \equiv (\nabla_{\lambda}u_{\rho} - \nabla_{\rho}u_{\lambda})/2$, the shear tensor $\sigma_{\lambda\rho} \equiv \nabla_{\langle\lambda}u_{\rho\rangle}$ and the expansion scalar $\theta \equiv \nabla_{\mu}u^{\mu}$, with u^{μ} being the fluid 4-velocity, $\nabla_{\mu} = \Delta_{\mu}^{\nu}\partial_{\nu}$ the projected spatial gradient. We used the notation $A^{\langle\mu\nu\rangle} \equiv \Delta_{\alpha\beta}^{\mu\nu}A^{\alpha\beta}$, with $\Delta_{\alpha\beta}^{\mu\nu} \equiv (\Delta_{\alpha}^{\mu}\Delta_{\beta}^{\nu} + \Delta_{\beta}^{\mu}\Delta_{\alpha}^{\nu} - 2/3\Delta^{\mu\nu}\Delta_{\alpha\beta})/2$, $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$, and $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The quantities multiplying each one of the terms are their corresponding transport coefficients, which are general functions of temperature (and chemical potential, if any) that should be extracted by matching fluid dynamics to the underlying microscopic theory, which here corresponds to the relativistic Boltzmann equation.

The above equations complement the conservation laws of energy-momentum. General formulas for the transport coefficients β_{Π} , $\delta_{\Pi\Pi}$, $\lambda_{\Pi\pi}$, β_{π} , $\delta_{\pi\pi}$, $\tau_{\pi\pi}$, and $\lambda_{\pi\Pi}$ were obtained in Refs. [42, 43], while expressions for φ_3 , φ_6 , and φ_7 were recently computed in Ref. [44]. However, such transport coefficients have been expressed formally, with most of them still not written in a convenient form to be implemented in fluid-dynamical simulations of heavy ion collisions or to provide a qualitative understanding of how these coefficients behave.

The coefficients $\delta_{\pi\pi}$, $\tau_{\pi\pi}$, and φ_7 do not vanish in the massless limit and, in this limit, can be trivially related to the shear viscosity relaxation time, τ_{π} , and the thermodynamic pressure, P_0 , [43, 44],

$$\delta_{\pi\pi} = \frac{4}{3}\tau_{\pi}, \quad \tau_{\pi\pi} = \frac{10}{7}\tau_{\pi}, \quad \varphi_7 = \frac{9}{70P_0}. \quad (3)$$

The formulas above are simple enough and can be used as estimates for $\delta_{\pi\pi}$, $\tau_{\pi\pi}$, and φ_7 in simulations of heavy ion collisions. As a matter of fact, the above expression for the transport coefficient $\delta_{\pi\pi}$ is already employed in all simulations of heavy ion collisions that include the shear-stress tensor. In Ref. [45] the coefficients $\tau_{\pi\pi}$ and φ_7 , and their corresponding nonlinear terms, were already included in fluid-dynamical simulations of heavy ion collisions. So far, similar expressions do not exist for the coefficients of bulk viscous pressure, which makes introducing bulk viscosity in heavy ion simulations in a complete and consistent way a more complicated task.

In this paper, we investigate the behavior of all coefficients associated to bulk viscous pressure and shear stress tensor when the mass is small (relative to the temperature), but is still finite. In this limit, we are able to express such coefficients in terms of the bulk and shear relaxation times as well as other thermodynamic variables. In summary, we find the following approximate expressions for the transport coefficients related to bulk viscous pressure,

$$\begin{aligned} \frac{\zeta}{\tau_{\Pi}} &= 14.55 \times \left(\frac{1}{3} - c_s^2\right)^2 (\varepsilon_0 + P_0) + \mathcal{O}(z^5), \\ \frac{\delta_{\Pi\Pi}}{\tau_{\Pi}} &= \frac{2}{3} + \mathcal{O}(z^2 \ln z), \\ \frac{\lambda_{\Pi\pi}}{\tau_{\Pi}} &= \frac{8}{5} \left(\frac{1}{3} - c_s^2\right) + \mathcal{O}(z^4), \end{aligned} \quad (4)$$

where $z \equiv m/T$, with m being the mass of the particle and T the temperature. The transport coefficients related to the shear-stress tensor become

$$\begin{aligned} \frac{\eta}{\tau_{\pi}} &= \frac{\varepsilon_0 + P_0}{5} + \mathcal{O}(z^2), \\ \frac{\delta_{\pi\pi}}{\tau_{\pi}} &= \frac{4}{3} + \mathcal{O}(z^2), \\ \frac{\tau_{\pi\pi}}{\tau_{\pi}} &= \frac{10}{7} + \mathcal{O}(z^2), \\ \frac{\lambda_{\pi\Pi}}{\tau_{\pi}} &= \frac{6}{5} + \mathcal{O}(z^2 \ln z), \end{aligned} \quad (5)$$

As mentioned, the expressions above are valid when the mass is small, i.e., $m/T \ll 1$. Such formulæ at least provide some intuition on how such coefficients behave and also on how they are related to other transport coefficients. In this sense, they can be useful to understand the parametric dependence of the fluid-dynamical transport coefficients on ε_0 , P_0 , c_s , τ_π , and τ_Π , making it less complicated to implement them in the description of the strongly interacting system created in heavy ion collisions.

This paper is organized as follows. Sections II, III, and IV summarize the basic steps required to derive relativistic fluid dynamics from the Boltzmann equation using the 14-moment approximation. Section V briefly explains the relaxation time approximation. In section VI we derive the main results of this paper, obtaining approximate expressions for the fluid-dynamical transport coefficients in the small, but finite, mass limit. In section VII we summarize our results and make our conclusions. Throughout this paper, we use natural units $\hbar = c = k_B = 1$.

II. FLUID DYNAMICS AND KINETIC THEORY

For the sake of simplicity, in this work we only consider the case of a single-component gas of classical particles. The starting point is the relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f], \quad (6)$$

where $C[f]$ is the collision term, and $k^\mu = (k^0 = \sqrt{\mathbf{k}^2 + m^2}, \mathbf{k})$ with m being the mass of the particle considered. We further employ the notation $f_{\mathbf{k}} \equiv f(x^\mu, k^\mu)$. For the purpose of this paper, it will not be necessary to specify the collision term.

The energy-momentum tensor $T^{\mu\nu}$ is expressed as a moment of the single particle distribution function

$$T^{\mu\nu} = \langle k^\mu k^\nu \rangle, \quad (7)$$

where we employ the following notation

$$\langle \dots \rangle \equiv \int dK (\dots) f_{\mathbf{k}}. \quad (8)$$

Above, $dK \equiv g d^3\mathbf{k} / [(2\pi)^3 k^0]$ is the Lorentz-invariant measure, with g being the appropriate degeneracy factor.

As usual, we define the fluid 4-velocity u^μ as an eigenvector of the energy-momentum tensor, $T^{\mu\nu} u_\mu = \varepsilon u^\nu$, where the eigenvalue ε is identified with the fluid local energy density [46]. Then, we can tensor-decompose the conserved current in terms of the four-velocity

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - \Delta^{\mu\nu} (P_0 + \Pi) + \pi^{\mu\nu}. \quad (9)$$

The energy density ε , the shear-stress tensor $\pi^{\mu\nu}$, and the sum of thermodynamic pressure, P_0 , and bulk viscous pressure, Π , are defined by

$$\varepsilon \equiv \langle (u_\mu k^\mu)^2 \rangle, \quad \pi^{\mu\nu} \equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle, \quad P_0 + \Pi \equiv -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle. \quad (10)$$

We define the local equilibrium distribution function as $f_{0\mathbf{k}} = \exp(-\beta_0 u_\mu k^\mu)$, where $\beta_0 = 1/T$ is the inverse temperature and we already assumed a vanishing chemical potential ($\mu = 0$). The temperature is defined from the following matching condition

$$\varepsilon \equiv \varepsilon_0 = \langle (u_\mu k^\mu)^2 \rangle_0, \quad (11)$$

where $\langle \dots \rangle_0 \equiv \int dK (\dots) f_{0\mathbf{k}}$.

The thermodynamic pressure and bulk viscous pressure are then defined by

$$P_0 = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0, \quad \Pi = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_\delta, \quad (12)$$

with $\langle \dots \rangle_\delta \equiv \langle \dots \rangle - \langle \dots \rangle_0$.

III. EQUATIONS OF MOTION FOR Π AND $\pi^{\mu\nu}$

The equations of motion for Π and $\pi^{\mu\nu}$ can be computed exactly for a single component gas using

$$\dot{\Pi} = -\frac{1}{3}m^2 \int dK \delta \dot{f}_{\mathbf{k}} , \quad (13)$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta \dot{f}_{\mathbf{k}} , \quad (14)$$

where we defined $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}}$. The comoving time derivative of δf , $\delta \dot{f} \equiv u^\mu \partial_\mu \delta f$, can then be simplified using the relativistic Boltzmann equation (6) in the form

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - \frac{1}{u_\mu k^\mu} k^\mu \nabla_\mu f_{\mathbf{k}} + \frac{1}{u_\mu k^\mu} C[f] . \quad (15)$$

After some algebra, one obtains the following equations,

$$\begin{aligned} \dot{\Pi} + C = & - \left[\left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0) - \frac{2}{9} (\varepsilon_0 - 3P_0) - \frac{m^4}{9} I_{-2,0} \right] \theta \\ & - (1 - c_s^2) \Pi \theta + \frac{m^4}{9} \rho_{-2} \theta + \left(\frac{1}{3} - c_s^2 \right) \pi^{\mu\nu} \sigma_{\mu\nu} + \frac{m^2}{3} \rho_{-2}^{\mu\nu} \sigma_{\mu\nu} , \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} + C^{\mu\nu} = & 2 \left[\frac{4}{5} P_0 + \frac{1}{15} (\varepsilon_0 - 3P_0) - \frac{m^4}{15} I_{-2,0} \right] \sigma^{\mu\nu} + 2 \pi^{\lambda\langle\mu} \omega_{\lambda}^{\nu\rangle} \\ & - \left(\frac{4}{3} \pi^{\mu\nu} + \frac{m^2}{3} \rho_{-2}^{\mu\nu} \right) \theta + \left(\frac{6}{5} \Pi - \frac{2}{15} m^4 \rho_{-2} \right) \sigma^{\mu\nu} \\ & - \left(\frac{10}{7} \pi^{\lambda\langle\mu} + \frac{4}{7} m^2 \rho_{-2}^{\lambda\langle\mu} \right) \sigma_{\lambda}^{\nu\rangle} . \end{aligned} \quad (17)$$

In the equations above, we have already neglected all terms related to particle diffusion. Also, irreducible moments of $\delta f_{\mathbf{k}}$ of rank higher than 2 were omitted, since such terms will not contribute to the fluid-dynamical equations and exactly vanish in the 14-moment approximation. The equations above are similar to those obtained in Refs. [42, 43], the only difference being that they were derived for a fixed chemical potential, $\mu = \dot{\mu} = 0$.

As in Ref. [43], we use the following notation for the collision terms,

$$\begin{aligned} C &= \frac{m^2}{3} \int dK (u_\alpha k^\alpha)^{-1} C[f] , \\ C^{\mu\nu} &= - \int dK (u_\alpha k^\alpha)^{-1} k^{\langle\mu} k^{\nu\rangle} C[f] . \end{aligned} \quad (18)$$

and for the irreducible moments of δf

$$\begin{aligned} \rho_n &= \langle (u_\alpha k^\alpha)^n \rangle_\delta , \\ \rho_n^\mu &= \langle (u_\alpha k^\alpha)^n k^{\langle\mu} \rangle , \\ \rho_n^{\mu\nu} &= \langle (u_\alpha k^\alpha)^n k^{\langle\mu} k^{\nu\rangle} \rangle . \end{aligned} \quad (19)$$

From equations (16) and (17), it is already possible to identify the ratio of bulk viscosity to bulk relaxation time, ζ/τ_Π , and of shear viscosity to shear relaxation time, η/τ_π , as the following thermodynamic quantities

$$\frac{\zeta}{\tau_\Pi} \equiv \beta_\Pi = \left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0) - \frac{2}{9} (\varepsilon_0 - 3P_0) - \frac{m^4}{9} I_{-2,0} , \quad (20)$$

$$\frac{\eta}{\tau_\pi} \equiv \beta_\pi = \frac{4}{5} P_0 + \frac{1}{15} (\varepsilon_0 - 3P_0) - \frac{m^4}{15} I_{-2,0} . \quad (21)$$

Here, we introduced the velocity of sound squared, $c_s^2 = dP_0/d\varepsilon_0$, which, at zero chemical potential, is given by

$$c_s^2 = \frac{\varepsilon_0 + P_0}{\beta_0 I_{30}} , \quad (22)$$

and made use of the thermodynamic functions I_{nq} ,

$$I_{nq} = \frac{1}{(2q+1)!!} \int dK (u_\mu k^\mu)^{n-2q} (-\Delta_{\mu\nu} k^\mu k^\nu)^q f_{0\mathbf{k}} . \quad (23)$$

Note that, in this notation, $I_{20} = \varepsilon_0$ and $I_{21} = P_0$. For a classical gas, the functions above satisfy the following rules

$$\begin{aligned} \beta_0 I_{nq} &= I_{n-1,q-1} + (n-2q) I_{n-1,q} , \\ I_{n+2,q} &= m^2 I_{nq} + (2q+3) I_{n+2,q+1} . \end{aligned} \quad (24)$$

These expressions are very convenient and can lead to rather useful relations, some of which being,

$$I_{10} = \beta_0 I_{21} , \quad I_{31} = \frac{\varepsilon_0 + P_0}{\beta_0} , \quad m^2 I_{00} = \varepsilon_0 - 3P_0 , \quad I_{42} = \frac{\varepsilon_0 + P_0}{\beta_0^2} . \quad (25)$$

The last expression is widely employed in simulations of heavy ion collisions when computing the nonequilibrium single-particle distribution function associated to the shear-stress tensor.

When written in terms of the velocity of sound, the expression found for ζ/τ_Π is exactly the same as the one found in Refs. [42, 47]. It should be noted that, even though the formula is the same, the behavior of the velocity of sound as a function of mass and temperature is not.

IV. 14-MOMENT APPROXIMATION

Obviously, equations (16) and (17) are not closed in terms of Π and $\pi^{\mu\nu}$ and, consequently, cannot be the final form of the fluid-dynamical equations. On the other hand, it is well known that a closed set of equations can be obtained from Eqs. (16) and (17) using the 14-moment approximation for the single-particle distribution function, first introduced by Israel and Stewart [41]. The 14-moment approximation dictates that

$$\frac{f_{\mathbf{k}} - f_{0\mathbf{k}}}{f_{0\mathbf{k}}} = \left[E_0 + B_0 m^2 + D_0 u_\mu k^\mu - 4B_0 (u_\mu k^\mu)^2 \right] \Pi + \lambda_n n_\alpha k^\alpha + B_2 \pi_{\alpha\beta} k^\alpha k^\beta . \quad (26)$$

The fifth term in the above expansion, $\lambda_n n_\alpha k^\alpha$, is related to particle diffusion and will be dropped, i.e., $\lambda_n = 0$. The coefficients E_0 , D_0 , B_0 , and B_2 are known functions of m , T , and $u_\mu k^\mu$ (and also of μ , when it is nonzero), and can be expressed in terms of the thermodynamic functions, I_{nq} , as

$$\begin{aligned} B_2 &= \frac{1}{2I_{42}} , \\ \frac{D_0}{3B_0} &= -4 \frac{I_{31}I_{20} - I_{41}I_{10}}{I_{30}I_{10} - I_{20}I_{20}} \equiv -C_2 , \\ \frac{E_0}{3B_0} &= m^2 + 4 \frac{I_{31}I_{30} - I_{41}I_{20}}{I_{30}I_{10} - I_{20}I_{20}} \equiv -C_1 , \\ B_0 &= \frac{1}{-3C_1 I_{21} - 3C_2 I_{31} - 3I_{41} - 5I_{42}} . \end{aligned} \quad (27)$$

In the 14-moment approximation (without particle diffusion), all moments of $\delta f_{\mathbf{k}}$ can be trivially written in terms of Π and $\pi^{\mu\nu}$,

$$\begin{aligned} \rho_{-n} &= \gamma_n^{(0)} \Pi , \\ \rho_{-n}^{\mu\nu} &= \gamma_n^{(2)} \pi^{\mu\nu} . \end{aligned} \quad (28)$$

The coefficients $\gamma_n^{(0)}$ and $\gamma_n^{(2)}$ are complicated functions of β_0 and m and read

$$\gamma_n^{(0)} = (E_0 + B_0 m^2) I_{-n,0} + D_0 I_{1-n,0} - 4B_0 I_{2-n,0} , \quad (29)$$

$$\gamma_n^{(2)} = \frac{I_{4-n,2}}{I_{42}} . \quad (30)$$

Note that only the coefficients $\gamma_n^{(0,2)}$ with $n = 2$ appear in the exact equations of motion for Π and $\pi^{\mu\nu}$.

Therefore, after making use of the 14-moment approximation, the equations of motion become

$$\begin{aligned} \dot{\Pi} + C = & -\frac{\zeta}{\tau_{\Pi}}\theta + \frac{m^2}{3}\nabla_{\mu}\rho_{-1}^{\mu} - \left[1 - c_s^2 - \frac{m^4}{9}\gamma_2^{(0)}\right]\Pi\theta \\ & + \left[\frac{1}{3} - c_s^2 + \frac{m^2}{3}\gamma_2^{(2)}\right]\pi^{\mu\nu}\sigma_{\mu\nu}, \end{aligned} \quad (31)$$

$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} + C^{\mu\nu} = & 2\frac{\eta}{\tau_{\pi}}\sigma^{\mu\nu} + 2\pi^{\lambda\langle\mu}\omega_{\lambda}^{\nu\rangle} - \left[\frac{10}{7} + \frac{4}{7}m^2\gamma_2^{(2)}\right]\pi^{\lambda\langle\mu}\sigma_{\lambda}^{\nu\rangle} \\ & - \left[\frac{4}{3} + \frac{m^2}{3}\gamma_2^{(2)}\right]\pi^{\mu\nu}\theta + \left[\frac{6}{5} - \frac{2}{15}m^4\gamma_2^{(0)}\right]\Pi\sigma^{\mu\nu}. \end{aligned} \quad (32)$$

The bulk and shear relaxation times, τ_{Π} and τ_{π} , respectively, come from applying the 14-moment approximation to the linearized collision term, leading to $C = \Pi/\tau_{\Pi}$ and $C^{\mu\nu} = \pi^{\mu\nu}/\tau_{\pi}$. The relaxation times, τ_{Π} and τ_{π} , arise from the collision term and, consequently, depend on the choice of cross section employed. All the other transport coefficients will depend on the cross sections only through the relaxation time, i.e., their ratio with the relaxation time is independent of the interaction strength. This will always be the case when the 14-moment approximation is employed.

By comparing with the fluid-dynamical equations shown in the introduction, Eqs. (1) and (2), we identify the most important transport coefficients. The coefficients appearing in the equation of motion for Π , besides ζ/τ_{Π} , are

$$\frac{\delta_{\Pi\Pi}}{\tau_{\Pi}} = 1 - c_s^2 - \frac{m^4}{9}\gamma_2^{(0)}, \quad (33)$$

$$\frac{\lambda_{\Pi\pi}}{\tau_{\Pi}} = \frac{1}{3} - c_s^2 + \frac{m^2}{3}\gamma_2^{(2)}, \quad (34)$$

while those appearing in the equation of motion for $\pi^{\mu\nu}$, besides η/τ_{π} , are found to be

$$\frac{\delta_{\pi\pi}}{\tau_{\pi}} = \frac{4}{3} + \frac{1}{3}m^2\gamma_2^{(2)}, \quad (35)$$

$$\frac{\tau_{\pi\pi}}{\tau_{\pi}} = \frac{10}{7} + \frac{4}{7}m^2\gamma_2^{(2)}, \quad (36)$$

$$\frac{\lambda_{\pi\Pi}}{\tau_{\pi}} = \frac{6}{5} - \frac{2}{15}m^4\gamma_2^{(0)}. \quad (37)$$

The remaining coefficients φ_i did not appear in the above equations since they originate solely from the nonlinear component of the collision term. In this paper, we will not consider these terms.

In the massless limit, only the coefficients β_{Π} and $\lambda_{\Pi\pi}$ vanish, i.e., $\beta_{\Pi} = \lambda_{\Pi\pi} = 0$. The remaining coefficients are finite in this limit and become

$$\lim_{m/T \rightarrow 0} \frac{\eta}{(\varepsilon_0 + P_0)\tau_{\pi}} = \frac{1}{5}, \quad (38)$$

$$\lim_{m/T \rightarrow 0} \frac{\delta_{\pi\pi}}{\tau_{\pi}} = \frac{4}{3}, \quad \lim_{m/T \rightarrow 0} \frac{\tau_{\pi\pi}}{\tau_{\pi}} = \frac{10}{7}, \quad (39)$$

$$\lim_{m/T \rightarrow 0} \frac{\lambda_{\pi\Pi}}{\tau_{\pi}} = \frac{6}{5}, \quad \lim_{m/T \rightarrow 0} \frac{\delta_{\Pi\Pi}}{\tau_{\Pi}} = \frac{2}{3}. \quad (40)$$

The coefficient $\delta_{\Pi\Pi}$ is the only one related to bulk viscous pressure that does not vanish. This is not a problem since this is the coefficient multiplying the term $\Pi\theta$ in Eq. (31) and Π itself vanishes as m^2 when $m/T \rightarrow 0$. Therefore, in the massless limit both sides of Eq. (31) will vanish, reducing it to the trivial equality $0 = 0$.

Note that both the shear and bulk coefficients depend on the same coefficients, $\gamma_2^{(0)}$ and $\gamma_2^{(2)}$. Therefore, the coefficients of bulk viscous pressure, ζ , τ_{Π} , $\delta_{\Pi\Pi}$, and $\lambda_{\Pi\pi}$, can be exactly related to those of the shear-stress tensor, η , τ_{π} , $\delta_{\pi\pi}$, $\tau_{\pi\pi}$, and $\lambda_{\pi\Pi}$, as follows

$$\frac{\zeta}{\tau_{\Pi}} = \frac{5}{3}\frac{\eta}{\tau_{\pi}} - c_s^2(\varepsilon_0 + P_0), \quad (41)$$

$$\frac{\delta_{\Pi\Pi}}{\tau_{\Pi}} = \frac{5}{6}\frac{\lambda_{\pi\Pi}}{\tau_{\pi}} - c_s^2, \quad (42)$$

$$\frac{\lambda_{\Pi\pi}}{\tau_{\Pi}} = \frac{1}{3} - c_s^2 + \frac{14}{9}\frac{\tau_{\pi\pi}}{\tau_{\pi}} - \frac{5}{3}\frac{\delta_{\pi\pi}}{\tau_{\pi}}. \quad (43)$$

V. RELAXATION TIME APPROXIMATION

In this section, we compute the collision integrals using the Bhatnagar-Gross-Krook (BGK) approximation [48], also known as the relaxation time approximation. In the BGK formalism the collision term is assumed to be proportional to the nonequilibrium single-particle momentum distribution function,

$$C[f] = -u_\mu k^\mu \frac{\delta f_{\mathbf{k}}}{\tau_R}, \quad (44)$$

where τ_R is an energy dependent relaxation time scale. Usually, this scale is parametrized to be of a certain power of $u_\mu k^\mu$,

$$\tau_R(u_\mu k^\mu) = \tau_{\text{mfp}} \left(\frac{u_\mu k^\mu}{T} \right)^\lambda, \quad (45)$$

with τ_{mfp} being a time scale proportional to the mean-free path of the system and λ a parameter that specifies the power of energy. For the sake of convenience, we fix $\lambda = 0$.

With this choice of approximation, the collision integrals appearing in Eqs. (31) and (32) can be easily solved, leading to

$$C = \frac{1}{\tau_{\text{mfp}}} \Pi, \quad C^{\mu\nu} = \frac{1}{\tau_{\text{mfp}}} \pi^{\mu\nu}. \quad (46)$$

Thus, we find the bulk and shear relaxation times to be equal, $\tau_\Pi = \tau_\pi = \tau_{\text{mfp}}$. Since the BGK approximation introduces only one time scale to describe the relaxation of the collision operator, it is natural to expect all the fluid-dynamical relaxation times to be proportional to this scale. However, one should note that both relaxation times are only equal because we set $\lambda = 0$. For other values of λ , $\tau_\Pi \neq \tau_\pi$, even though they would both remain proportional to τ_{mfp} .

VI. EXPANSION FOR SMALL MASSES

Using the properties of modified Bessel functions of the second kind, the transport coefficients derived in the previous section can be expanded in a series of $z = m/T$. The corresponding Bessel function can be written in an integral form as

$$K_n(z) = \frac{\Gamma(1/2)}{\Gamma(n+1/2)} \left(\frac{z}{2} \right)^n \int_0^\infty d\theta (\sinh \theta)^{2n} \exp[-z \cosh(\theta)], \quad (47)$$

where Γ is the gamma function [49].

The thermodynamic integrals I_{nq} previously introduced can be cast in a similar form,

$$I_{nq}(T, z) = \frac{T^{n+2}}{(2q+1)!!} \frac{z^{n+2}}{2\pi^2} \int_0^\infty d\theta (\cosh \theta)^{n-2q} (\sinh \theta)^{2q+2} \exp(-z \cosh \theta). \quad (48)$$

It is straightforward to express all thermodynamic integrals in terms of Bessel functions, its derivatives, and/or its integrals. For example, the thermodynamic pressure and energy density can be written in terms of K_1 and K_2 as

$$P_0(T, z) = I_{21}(T, z) = \frac{1}{2} P_0(T, 0) z^2 K_2(z), \quad (49)$$

$$\varepsilon_0(T, z) = I_{20}(T, z) = 3P_0(T, z) + \frac{1}{2} P_0(T, 0) z^3 K_1(z), \quad (50)$$

while the particle number density as

$$n_0(T, z) = I_{10}(T, z) = \frac{1}{2} n_0(T, 0) \left[z K_1(z) - z^2 \frac{d}{dz} K_1(z) \right]. \quad (51)$$

The Bessel function $K_n(z)$ can be expanded in terms of z as [49]

$$\begin{aligned} K_n(z) &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{z}{2} \right)^{2k-n} \\ &+ (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{k! (n+k)!} \left(\frac{z}{2} \right)^{n+2k} \left[\ln \frac{z}{2} - \frac{1}{2} \psi(k+1) - \frac{1}{2} \psi(n+k+1) \right], \end{aligned} \quad (52)$$

with ψ being the digamma function. Using this series expansion of $K_n(z)$, one can obtain the dominant terms when $z = m/T \ll 1$ for several thermodynamic quantities. For the transport coefficients of bulk viscous pressure, the most relevant quantities are

$$\begin{aligned}
\frac{\varepsilon_0(T, z) - 3P_0(T, z)}{P_0(T, 0)} &= \frac{1}{2}z^2 + \frac{1}{8}z^4(2\ln z + 2\gamma - 1 - \ln 4) + \mathcal{O}(z^6 \ln z) , \\
\frac{1}{3} - c_s^2 &= \frac{1}{36}z^2 - \frac{5}{864}z^4 + \mathcal{O}(z^6 \ln z) , \\
2\pi^2 I_{-20} &= \ln z + a + \mathcal{O}(z) , \\
\frac{2\pi^2}{T} I_{-10} &= 1 - b z + \mathcal{O}(z^2 \ln z) , \\
m^2 \gamma_2^{(2)} &= \frac{1}{20}z^2 + \mathcal{O}(z^4) .
\end{aligned} \tag{53}$$

where a and b are integration constants that can be identified to be $a = 0.86$ and $b = 1.6$. Furthermore, the expansion coefficients appearing in δf can be shown to be all of order $\mathcal{O}(z^{-2})$,

$$B_0(z) = \frac{3}{4}z^{-2} + \mathcal{O}(1) , \quad D_0(z) = 24z^{-2} + \mathcal{O}(1) , \quad E_0(z) = -36z^{-2} + \mathcal{O}(1) . \tag{54}$$

Substituting Eqs. (53) and (54) into Eq. (29) one then obtains

$$m^4 \gamma_2^{(0)} = 18z^2 \ln z + \mathcal{O}(z^2) . \tag{55}$$

Next, using Eqs. (53), (54), and (55) we can calculate the lowest order z dependence of the transport coefficients that vanish when $z = 0$, i.e., the coefficients ζ/τ_Π and $\lambda_{\Pi\pi}/\tau_\Pi$. We find that

$$\frac{\zeta}{\tau_\Pi} \sim \mathcal{O}(z^4) , \quad \frac{\lambda_{\Pi\pi}}{\tau_\Pi} \sim \mathcal{O}(z^2) . \tag{56}$$

As already mentioned, the remaining coefficients do not vanish and, consequently, $\beta_\pi/(\varepsilon_0 + P_0) \sim \delta_{\pi\pi}/\tau_\pi \sim \tau_{\pi\pi}/\tau_\pi \sim \delta_{\Pi\Pi}/\tau_\Pi \sim \mathcal{O}(1)$. It is particularly interesting that the ratio ζ/τ_Π behaves as $\mathcal{O}(z^4)$, since $(1/3 - c_s^2)$ and $(\varepsilon - 3P_0)/P_0$ behave as $\mathcal{O}(z^2)$ and $z^4 I_{-2,0}$ behaves as $\mathcal{O}(z^4 \ln z)$. However, as it turns out, all the terms that contribute to order $\mathcal{O}(z^2)$, $\mathcal{O}(z^2 \ln z)$, and $\mathcal{O}(z^4 \ln z)$ simply cancel in the expression for ζ/τ_Π , leaving a lowest order contribution of $\mathcal{O}(z^4)$, as already stated. Such behavior is not obvious and could not be inferred simply by looking at Eq. (20).

As shown in Eq. (53), $1/3 - c_s^2$ is of order $\mathcal{O}(z^2)$. Therefore, if we are only interested in the transport coefficients up to order $\mathcal{O}(z^4)$, all terms proportional to z^2 can be replaced by $1/3 - c_s^2$. In this case, ζ/τ_Π will be proportional $(1/3 - c_s^2)^2$ while $\lambda_{\Pi\pi}/\tau_\Pi$ will just be proportional to $(1/3 - c_s^2)$. More explicitly, it is possible to prove that

$$\frac{\zeta}{\tau_\Pi} = 14.55 \times \left(\frac{1}{3} - c_s^2\right)^2 (\varepsilon_0 + P_0) + \mathcal{O}(z^5) , \tag{57}$$

$$\frac{\lambda_{\Pi\pi}}{\tau_\Pi} = \frac{8}{5} \left(\frac{1}{3} - c_s^2\right) + \mathcal{O}(z^4) . \tag{58}$$

Note that, at least when $z = m/T$ is small, $\lambda_{\Pi\pi}$, which describes the coupling of bulk viscous pressure to shear-stress tensor, is much larger than the bulk viscosity, ζ , itself. This might indicate that the coupling to shear-stress tensor ($\lambda_{\Pi\pi} \pi^{\mu\nu} \sigma_{\mu\nu}$) can give a contribution to the build up of the bulk viscous pressure that is comparable to the one originating from the Navier-Stokes term ($-\zeta\theta$). In this sense, such coupling term should be included when investigated the effects of bulk viscous pressure.

Using the relaxation time approximation, as introduced in the previous section, one can use Eqs. (38) and (57) to relate ζ and η ,

$$\zeta = 72.75 \eta \left(\frac{1}{3} - c_s^2\right)^2 + \mathcal{O}(z^5) . \tag{59}$$

Qualitatively, this expression is similar to the known relation, $\zeta = 15\eta (1/3 - c_s^2)^2$, found by Weinberg more than 40 years ago [50]. However, quantitatively we obtain a proportionally factor relating ζ and $\eta (1/3 - c_s^2)^2$ that is

almost five times larger than the one displayed in Ref. [50]. This is not a problem since, motivated by applications to cosmology, Weinberg derived his formula for a material medium, with very short mean-free times, interacting with radiation. This is, of course, a very different system than the one considered in this paper and there is no reason to expect our calculation to coincide with Weinberg's.

Note that the temperature dependence of ζ/τ_Π is quite different from the temperature dependence of η/τ_π . While the latter is just proportional (in the ultrarelativistic limit) to $(\varepsilon_0 + P_0)$, the former displays a more complicated behavior, being proportional to $(1/3 - c_s^2)^2(\varepsilon_0 + P_0)$. In this sense, the approximation $\zeta/\tau_\Pi \sim (\varepsilon_0 + P_0)$ is not so good, even though it naively appears to be the simplest parametric expression for ζ/τ_Π that guarantees causality [51]. On the other hand, for the case of shear viscosity $\eta/\tau_\pi \sim (\varepsilon_0 + P_0)$ works very well, at least in the $m/T < 1$ limit.

All the other transport coefficients behave as a function of $z = m/T$ in the following way

$$\begin{aligned} \frac{\beta_\pi}{\varepsilon_0 + P_0} &= \frac{1}{5} - \frac{1}{60}z^2 + \mathcal{O}(z^4 \ln z), \\ \frac{\delta_{\pi\pi}}{\tau_\pi} &= \frac{4}{3} + \frac{1}{60}z^2 + \mathcal{O}(z^4 \ln z), \\ \frac{\tau_{\pi\pi}}{\tau_\pi} &= \frac{10}{7} + \frac{1}{35}z^2 + \mathcal{O}(z^4 \ln z), \\ \frac{\delta_{\Pi\Pi}}{\tau_\Pi} &= \frac{2}{3} - (2 \ln z + 3.015)z^2 + \mathcal{O}(z^3), \\ \frac{\lambda_{\pi\Pi}}{\tau_\pi} &= \frac{6}{5} - \left(\frac{12}{5} \ln z + 3.65\right)z^2 + \mathcal{O}(z^3). \end{aligned} \quad (60)$$

Most of them are constant up to order $\mathcal{O}(z^2)$. The coefficients $\delta_{\Pi\Pi}/\tau_\Pi$ and $\lambda_{\pi\Pi}/\tau_\pi$ are constant up to $\mathcal{O}(z^2 \ln z)$, displaying a stronger dependence on m/T . Overall, we find it to be a good approximation to set all these terms as constants. Such constant values can be thought of as lower bounds for these coefficients, since most of them will increase with increasing m/T .

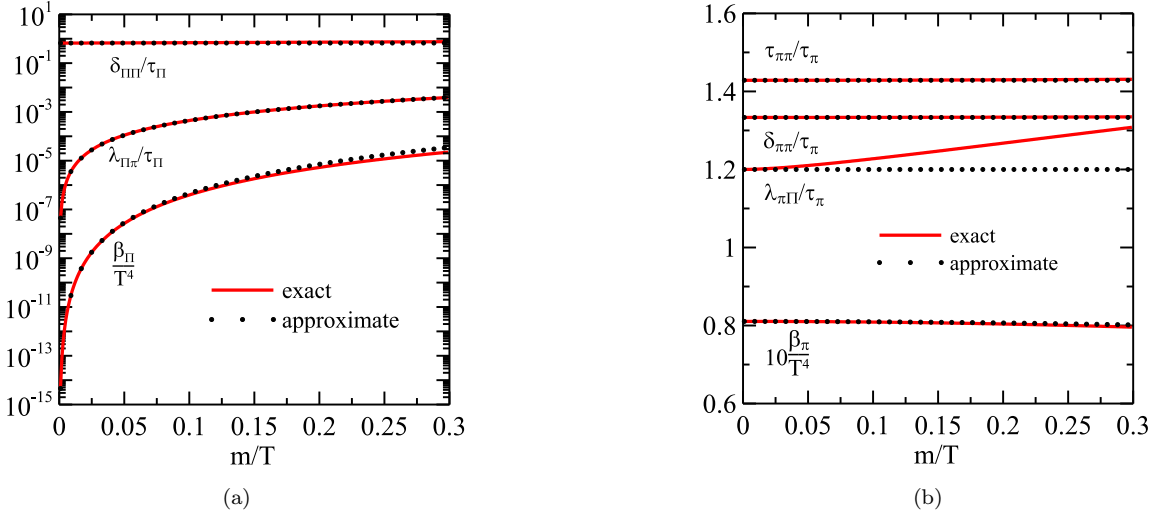


FIG. 1: (Color online) In this figure we compare the approximate expressions (circles) of the transport coefficients obtained in the main text with their exact values (solid red lines), as a function of $mass/T$. The left panel contains the transport coefficients appearing in the equations of motion for the bulk viscous pressure while the right panel contains the transport coefficients contained in the equations of motion for the shear-stress tensor.

The approximate expressions we found for the transport coefficients $\lambda_{\pi\Pi}/\tau_\pi$ and $\lambda_{\Pi\pi}/\tau_\Pi$, which describe the coupling between shear-stress tensor and bulk viscous pressure, were the first to relate them to well-known thermodynamic quantities. In this way, we can understand in a qualitative way, at least in the ultrarelativistic limit, how such transport coefficients vary as we change the velocity of sound. This simple behavior was unknown up to now.

The transport coefficient $\delta_{\Pi\Pi}/\tau_\Pi$ is somewhat better understood. The term $\delta_{\Pi\Pi}\Pi\theta$, which appears in the equation of motion for the bulk viscous pressure, is also commonly written in a different, but equivalent, way as

$$\frac{1}{2}\Pi\frac{\zeta T}{\tau_\Pi}\partial_\mu\left(\frac{\tau_\Pi}{\zeta T}u^\mu\right). \quad (61)$$

Such term usually appears when deriving the equations of motion for Π phenomenologically, via the second law of thermodynamics [52]. If we use the expression ζ/τ_Π found in this paper and substitute it in Eq. (61), it is possible to show that this term, and the transport coefficient multiplying it, is also equivalent to the ones obtained in this paper. However, if a different ζ/τ_Π is employed, one would then find that Eq. (61) would lead to a different and wrong expression for $\delta_{\Pi\Pi}/\tau_\Pi$.

In Fig. 1, we compare the exact formula for these coefficients with the approximate expressions just introduced in Eqs. (57), (58), and (60). The approximate expressions for the coefficients β_Π and $\lambda_{\Pi\pi}/\tau_\Pi$ are shown up to $\mathcal{O}(z^5)$ and $\mathcal{O}(z^4)$, respectively, while only the constant parts (in z) of the remaining coefficients are displayed. We see that the agreement between the exact expressions and their approximate counterparts is reasonable up to $z \approx 0.3$, where largest difference observed could reach 20% for $\delta_{\Pi\Pi}/\tau_\Pi$ and $\lambda_{\pi\Pi}/\tau_\pi$.

More precise, yet more complicated, expressions for these transport coefficients can also be obtained. For the sake of completeness, we list them below,

$$\begin{aligned}
\frac{\beta_\pi}{\varepsilon_0 + P_0} &= \frac{1}{5} - \frac{3}{5} \left(\frac{1}{3} - c_s^2 \right) + \mathcal{O}(z^4 \ln z) , \\
\frac{\delta_{\pi\pi}}{\tau_\pi} &= \frac{4}{3} + \frac{3}{5} \left(\frac{1}{3} - c_s^2 \right) + \mathcal{O}(z^4 \ln z) , \\
\frac{\tau_{\pi\pi}}{\tau_\pi} &= \frac{10}{7} + \frac{36}{35} \left(\frac{1}{3} - c_s^2 \right) + \mathcal{O}(z^4 \ln z) , \\
\frac{\delta_{\Pi\Pi}}{\tau_\Pi} &= \frac{2}{3} + \left[4 - \frac{8}{9} \frac{\varepsilon_0 - 3P_0}{\left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0)} \right] - 108.89 \times \left(\frac{1}{3} - c_s^2 \right) + \mathcal{O}(z^3) , \\
\frac{\lambda_{\pi\Pi}}{\tau_\pi} &= \frac{6}{5} + \frac{24}{5} \left[1 - \frac{2}{9} \frac{\varepsilon_0 - 3P_0}{\left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0)} \right] - 127.02 \times \left(\frac{1}{3} - c_s^2 \right) + \mathcal{O}(z^3) .
\end{aligned} \tag{62}$$

In the formulas above, the z^2 and $z^2 \ln z$ dependences were removed using the following relations

$$\begin{aligned}
\frac{1}{3} - c_s^2 &= \frac{1}{36} z^2 + \mathcal{O}(z^4) , \\
\frac{\left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0) - \frac{2}{9} (\varepsilon_0 - 3P_0)}{\left(\frac{1}{3} - c_s^2 \right) (\varepsilon_0 + P_0)} &= -\frac{1}{2} z^2 \ln z - 0.02536 \times z^2 + \mathcal{O}(z^4 \ln z) .
\end{aligned} \tag{63}$$

Most of these formulas are accurate up to order $\mathcal{O}(z^4 \ln z)$. Only the expressions for $\delta_{\Pi\Pi}/\tau_\Pi$ and $\lambda_{\pi\Pi}/\tau_\pi$, which always exhibit a stronger mass dependence, are accurate only up to $\mathcal{O}(z^3)$.

It is important to remark that one can also find an approximate expression for the momentum distribution function by expanding it in powers of z . Using that

$$\begin{aligned}
B_0 &= \frac{3}{4} \frac{1}{z^2} + \mathcal{O}(1) , \\
D_0 &= 24 \frac{1}{z^2} + \mathcal{O}(1) , \\
E_0 &= -36 \frac{1}{z^2} + \mathcal{O}(1) ,
\end{aligned} \tag{64}$$

and replacing these approximate expressions into Eq. (26), one can show that

$$\frac{f_{\mathbf{k}} - f_{0\mathbf{k}}}{f_{0\mathbf{k}}} \approx \frac{4}{z^2} \left[-36 + 24 \frac{1}{T} u_\mu k^\mu - 3 \frac{1}{T^2} (u_\mu k^\mu)^2 \right] \frac{\Pi}{\varepsilon_0 + P_0} + \mathcal{O}(1) .$$

This is an approximate and more convenient form for the single particle distribution function, at least when the mass is relatively small compared to the temperature.

VII. CONCLUSION

In this paper we computed the transport coefficients that appear in the fluid-dynamical equations for the bulk viscous pressure and shear-stress tensor using the 14-moment approximation for a single component relativistic dilute

gas. The main results of this paper were computed in the limit of small, but finite, mass where we showed it was possible to express all the fluid-dynamical transport coefficients in terms of known thermodynamic quantities, such as the thermodynamic pressure, energy density, and the velocity of sound.

We explicitly demonstrated that the ratio of bulk viscosity to bulk relaxation time behaves very differently, as a function of temperature, than the ratio of shear viscosity to shear relaxation time. It was found that the well known expression found by Weinberg, $\zeta = 15\eta(1/3 - c_s^2)^2$, for a system of radiation coupled to matter, does not apply in general. In the limit of small m/T , one does in fact find that $\zeta \sim \eta(1/3 - c_s^2)^2$, but the proportionality factor does not have to be 15. Using the relaxation time approximation, we actually found it to be ~ 75 . When the mass is not small, even the proportionality $\zeta \sim \eta(1/3 - c_s^2)^2$ does not need to hold.

We further explicitly computed, for the first time, the transport coefficients that couple the bulk viscous pressure to the shear-stress tensor and vice versa. The coefficient that couples bulk viscous pressure to shear-stress tensor is found to be orders of magnitude larger than the bulk viscosity itself. This indicates that the bulk viscous pressure might be produced in heavy ion collisions more due to its coupling to shear, than due to the expansion rate of the system. So far, no simulations of heavy ion collisions have ever taken into account the coupling that may exist between different dissipative currents. For bulk viscous pressure, neglecting the coupling to the shear-stress tensor might be a particularly bad approximation.

All calculations in this paper were performed for a single component system and neglect the effect of baryon chemical potential. This can be considered a good feature, if one is just interested in the qualitative behavior of the fluid-dynamical transport coefficients; especially on how they depend on certain thermodynamic variables. In a future work, we plan to extend the findings of this paper, considering a multi-component system and a nonvanishing chemical potential. Then we shall have more quantitative results and can check how well the qualitative expressions found in this paper can describe more realistic systems.

VIII. ACKNOWLEDGMENTS

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